Probability and Applied Statistics

Formula Sheet

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# 1 What Is Statistics?

## Definition 1.1: Mean

The *mean* of a sample of measured responses is given by

The corresponding population mean is denoted .

## Definition 1.2: Variance

The *variance* of a sample of measurements is the sum of the square of the differences between the measurements and their mean, divided by . Symbolically, the sample variance is

The corresponding population variance is denoted by the symbol .

## Definition 1.3: Standard Deviation

The *standard deviation* of a sample of measurements is the positive square root of the variance; that is,

The corresponding population standard deviation is denoted by .

# 2 Probability

## Extra 2.1: Union

The union of and , denoted by , is the set of all points in or or both.

## Extra 2.2: Intersection

The intersection of and , denoted by or by , is the set of all points in both and .

## Extra 2.3: Complement

If is a subset of , then the complement of , denoted by , is the set of points that are in but not in . Note that .

## Extra 2.4: Mutually Exclusive

Two sets, and , are said to be *disjoint*, or *mutually exclusive*, if . That is, mutually exclusive sets have no points in common. Note that, for any set , and are mutually exclusive.

## Extra 2.5: Distributive Laws

## Extra 2.6: De Morgan’s Laws

## Definition 2.1: Experiment

An *experiment* is the process by which an observation is made.

## Definition 2.2: Simple Event

A *simple event* is an event that CANNOT be decomposed. Each simple event corresponds to one and only one *sample point*. The letter *E* with a subscript will be used to denote a simple event or the corresponding sample point. Note: An event that CAN be decomposed into other events is called a *compound event*.

## Definition 2.3: Sample Space

The *sample space* associated with an experiment is the set consisting of all possible sample points. A sample space will be denoted by .

## Definition 2.4: Discrete Sample Space

A *discrete sample space* is one that contains either a finite or a countable number of distinct sample points.

## Definition 2.5: Event

An *event* in a discrete sample space is a collection of sample point­s—that is, any subset of *S*.

## Definition 2.6: Probability

Suppose is a sample space associated with an experiment. To every event in ( is a subset of ), we assign a number, , called the *probability* of , so that the following axioms hold:

Axiom 1:

Axiom 2:

Axiom 3: If form a sequence of pairwise mutually exclusive events in S (that is, if ), then

## Theorem 2.1: Rule

With elements and elements , it is possible to form pairs containing one element from each group.

## Definition 2.7: Permutation

An ordered arrangement of distinct objects is called a *permutation*. The number of ways of ordering distinct objects taken at a time will be designated by the symbol .

## Theorem 2.2: Number of Distinct Arrangements

## Theorem 2.3: Multinomial Coefficient

The number of ways of partitioning distinct objects into distinct groups containing objects, respectively, where each object appears in exactly one group and , is

## Definition 2.8: Combination

The number of *combinations* of objects taken at a time is the number of subsets, each of size , that can be formed from the objects. This number will be denoted by or .

## Theorem 2.4: Number of Unordered Subsets

The number of unordered subsets of size *r* chosen (without replacement) from *n* available objects is

## Definition 2.9: Conditional Probability

The *conditional probability* of an event , given that an event has occurred, is equal to

provided . [The symbol is read “probability of given .”]

## Definition 2.10: Independent Events

Two events and are said to be *independent* if any one of the following holds:

Otherwise, the events are said to be *dependent*.

## Theorem 2.5: The Multiplicative Laws of Probability

The probability of the intersection of two events and is

If and are independent, then

## Theorem 2.6: The Additive Law of Probability

The probability of the union of two events and is

If and are mutually exclusive events, and

## Theorem 2.7: Complement Rule

If is an event, then

## Definition 2.11: Partition

For some positive integer , let the sets be such that

1.

2. , for

Then the collection of sets is said to be a *partition* of .

## Theorem 2.8: The Law of Total Probability

Assume that is a partition of such that , for . Then for any event

## Theorem 2.9: Bayes’ Rule

Assume that is a partition of such that , for . Then

## Definition 2.12: Random Variable

A *random variable* is a real-valued function for which the domain is a sample space.

## Definition 2.13: Random Sample

Let and represent the numbers of elements in the population and sample, respectively. If the sampling is conducted in such a way that each of the samples has an equal probability of being selected, the sampling is said to be random, and the result is said to be a *random sample*.

# 3 Discrete Random Variables and Their Probability Distributions

## Definition 3.1: Discrete

A random variable is said to be *discrete* if it can assume only a finite or countably infinite number of distinct values.

## Definition 3.2: Probability Function for Y

The probability that takes on the value , , is defined as the *sum of the probabilities of all sample points in* that are assigned the value . We will sometimes denote by .

## Definition 3.3: Probability Distribution

The *probability distribution* for a discrete variable can be represented by a formula, a table, or a graph that provides for all .

## Theorem 3.1: Properties of Discrete Probability Distribution

For any discrete probability distribution, the following must be true:

1. , for all .

2. , where the summation is over all values of with nonzero probability.

## Definition 3.4: Expected Value of a Discrete Random Variable

Let be a discrete random variable with the probability function . Then the *expected value* of , , is defined to be

## Theorem 3.2: Expected Value of a Real-Valued Function of a Discrete Random Variable

Let be a discrete random variable with probability function and be a real-valued function of . Then the expected value of is given by

## Definition 3.5: Standard Deviation of a Discrete Random Variable

If is a random variable with mean , the variance of a random variable is defined to be the expected value of . That is,

The *standard deviation* of is the positive square root of .

## Theorem 3.3: Expected Value of Nonrandom Quantity

Let be a discrete random variable with probability function and be a constant. Then .

## Theorem 3.4: Expected Value of the Product of a Constant and a Function of a Random Variable

Let be a discrete random variable with probability function , be a function of , and be a constant. Then

## Theorem 3.5: Expected Value of a Sum of Functions of a Random Variable

Let be a discrete random variable with probability function and be functions of . Then

## Theorem 3.6: Variance of a Discrete Random Variable

Let be a discrete random variable with probability function and mean ; then

## Definition 3.6: Binomial Experiment

A *binomial experiment* possesses the following properties:

1. The experiment consists of a fixed number, , or identical trials.

2. Each trial results in one of two outcomes: success, , or failure, .

3. The probability of success on a single trial is equal to some value and remains the same from trial to trial. The probability of a failure is equal to .

4. The trials are independent.

5. The random variable of interest is , the number of successes observed during the trials.

## Definition 3.7: Binomial Distribution

A random variable is said to have a *binomial distribution* based on trials with success probability if and only if

and

## Theorem 3.7: Mean and Variance of a Binomial Random Variable

Let be a binomial random variable based on trials and success probability . Then

and

## Definition 3.8: Geometric Probability Distribution

A random variable is said to have a *geometric probability distribution* if and only if

## Theorem 3.8: Mean of a Geometric Random Variable

If is a random variable with a geometric distribution,

and

## Definition 3.9: Negative Binomial Probability Distribution

A random variable is said to have a *negative binomial probability distribution* if and only if

## Theorem 3.9: Expected Value and Variance of a Negative Binomial Distribution

If is a random variable with a negative binomial distribution,

and

## Definition 3.10: Hypergeometric Probability Distribution

A random variable is said to have a *hypergeometric probability distribution* if and only if

,

Where is an integer , subject to the restrictions and .

## Theorem 3.10: Expected Value and Variance of a Hypergeometric Distribution

If is a random variable with a hypergeometric distribution,

and .

## Definition 3.11: Poisson Probability Distribution

A random variable is said to have a *Poisson probability distribution* if and only if

.

## Theorem 3.11: Expected Value and Variance of a Poisson Distribution

If is a random variable possessing a Poisson distribution with parameter , then

and .

## Definition 3.12: Moment of a Random Variable Taken About the Origin

The th *moment of a random variable* *taken about the origin* is defined to be and is denoted by .

## Definition 3.13: Moment of a Random Variable Taken About its Mean

The th *moment of a random variable* *taken about its mean*, or the th *central moment of* , is defined to be and is denoted by .

## Definition 3.14: Moment-Generating Function

The *moment-generating function* *for a random variable* is defined to be . We say that a moment-generating function for exists if there exists a positive constant such that is finite for .

## Theorem 3.12: The kth Derivative of m(t)

If exists, then for any positive integer ,

.

In other words, if you find the th derivative of with respect to and then set , the result will be .

## Definition 3.15: Probability Generating Function

Let be an integer-valued random variable for which , where . The *probability-generating function* for is defined to be

for all values of such that is finite.

## Definition 3.16: The kth Factorial Moment

The th *factorial moment* for a random variable is defined to be

where is a positive integer.

## Theorem 3.13: The kth Factorial Moment of an Integer-valued Random Variable

If is a probability-generating function for an integer-valued random variable, , then the th factorial moment of is given by

.

## Theorem 3.14: Tchebysheff’s Theorem

Let be a random variable with mean and finite variance . Then, for any constant ,

or .

# 4 Continuous Variables and Their Probability Distributions

## Definition 4.1: Distribution Function

Let denote any random variable. The *distribution function* of , denoted by , is such that for .

## Theorem 4.1: Properties of a Distribution Function

If is a distribution function, then

1. .
2. .
3. is a nondecreasing function of . [If and are *any* values such that , then .]

## Definition 4.2: Continuous Random Variable

A random variable with distribution function is said to be *continuous* if is continuous, for .

## Definition 4.3: Probability Density Function

Let be the distribution function for a continuous random variable . Then , given by

wherever the derivative exists, is called the *probability density function* for the random variable .

## Theorem 4.2: Properties of a Density Function

If is a density function for a continuous random variable, then

1. for all , .
2. .

## Definition 4.4: The pth Quantile

Let denote any random variable. If , the *th quantile* of , denoted by , is the smallest value such that . If is continuous, is the smallest value such that

. Some prefer to call the 100 *th percentile* of .

## Theorem 4.3: Finding Probability Given a Density Function

If the random variable has density function and , then the probability that falls in the interval is

.

## Definition 4.5: Expected Value of a Continuous Random Variable

The expected value of a continuous random variable is

,

provided that the integral exists.

## Theorem 4.4: Expected Value of a Function of a Continuous Random Variable

Let be a function of ; then the expected value of is given by

,

provided that the integral exists.

## Theorem 4.5: Expected Values of More Important Functions of a Continuous Random Variable

Let be a constant and let be functions of a continuous random variable . Then the following results hold:

1. .
2. .
3. .

## Definition 4.6: Uniform Probability Distribution

If , a random variable is said to have a continuous *uniform probability distribution* on the interval if and only if the density function of is

## Definition 4.7: Parameters of a Density Function

The constants that determine the specific form of a density function are called *parameters* of the density function.

## Theorem 4.6: Expected Value and Variance of a Uniform Probability Distribution

If and is a random variable uniformly distributed on the interval , then

and .

## Definition 4.8: Normal Probability Distribution

A random variable is said to have a *normal probability distribution* if and only if, for and

, the density function of is

.

## Theorem 4.7: Expected Value and Variance of a Normal Distribution

If is a normally distributed random variable with parameters and , then

and .

## Definition 4.9: Gamma Distribution

A random variable is said to have a *gamma distribution* with parameters and if and only if the density function of is

where

.

## Theorem 4.8: Expected Value and Variance of a Gamma Distribution

If has a gamma distribution with parameters and , then

and .

## Definition 4.10: Chi-Square Distribution

Let be a positive integer. A random variable is said to have a *chi-square distribution* with degrees of freedom if and only if is a gamma-distributed random variable with parameters and

.

## Theorem 4.9: Expected Value and Variance of a Chi-Square Distribution

If is a chi-square random variable with degrees of freedom, then

and .

## Definition 4.11: Exponential Distribution

A random variable is said to have an *exponential distribution* with parameter if and only if the density function of is

## Theorem 4.10: Expected Value and Variance of an Exponential Distribution

If is an exponential random variable with parameter , then

and .

## Definition 4.12: Beta Probability Distribution

A random variable is said to have a *beta probability distribution* with parameters and if and only if the density function of is

where

.

## Theorem 4.11: Expected Value and Variance of a Beta Distribution

If is a beta-distributed random variable with parameters and , then

and .

## Definition 4.13: Moment of a Continuous Random Variable

If is a continuous random variable, then the th *moment about the origin* is given by

The th *moment about the mean*, or the th *central moment*, is given by

## Definition 4.14: Moment-Generating Function of a Continuous Random Variable

If is a continuous random variable, then the *moment-generating function of* is given by

.

The moment-generating function is said to exist if there exists a constant such that is finite for .

## Theorem 4.12: Moment-Generating Function of a Function of a Continuous Random Variable

Let be a random variable with density function and be a function of . Then the moment-generating function for is

.

## Theorem 4.13: Tchebysheff’s Theorem Restated

Let be a random variable with finite mean and variance . Then for any ,

or .

## Definition 4.15: Expected Value of a Mixed Distribution

Let have the mixed distribution

and suppose that is a discrete random variable with distribution function and that is a continuous random variable with distribution function . Let denote a function of . Then

# 5 Multivariate Probability Distributions

## Definition 5.1: Joint Probability Function

Let and be discrete random variables. The *joint* (or bivariate) *probability function* for and is given by

.

## Theorem 5.1: Joint Probability Function Properties

If and are discrete random variables with joint probability function , then

1. for all .
2. , where the sum is over all values that are assigned nonzero probabilities.

## Definition 5.2: Joint Distribution Function

For any random variables and , the joint (bivariate) distribution function is

.

## Definition 5.3: Joint Probability Density Function

Let and be continuous random variables with joint distribution function . If there exists a nonnegative function , such that

,

for all , then and are said to be jointly continuous random variables. The function is called the *joint probability density function*.

## Theorem 5.2: Joint Distribution Function Properties

If and are random variables with joint distribution function , then

1. .
2. .
3. If and , then

## Theorem 5.3: Joint Density Function Properties

If and are jointly continuous random variables with a joint density function given by , then

1. for all .

## Definition 5.4: Marginal Probability Functions and Marginal Density Functions

1. Let and be jointly discrete random variables with probability function . Then the *marginal probability functions* of and , respectively, are given by

and .

1. Let and be jointly continuous random variables with joint density function . Then the *marginal density functions* of and , respectively, are given by

and .

## Definition 5.5: Conditional Discrete Probability Function

If and are jointly discrete random variables with joint probability function and marginal probability functions and , respectively, then the *conditional discrete probability function* of given is

provided that

## Definition 5.6: Conditional Distribution Function

If and are jointly continuous random variables with joint density function , then the conditional distribution function of given is

.

## Definition 5.7: Conditional Density Functions

Let and be jointly continuous random variables with joint density and marginal densities and , respectively. For any such that , the conditional density of given is given by

and, for any such that , the conditional density of given is given by